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Rational Approximations with Real Poles to e^{-x} and x^n

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INTRODUCTION

It is possible, and in some cases desirable, to uniformly approximate e^{-x} on $[0, \infty)$ by rational functions with real negative poles. Saff *et al.* [10] examined this question in 1976 and showed that asymptotically the nth root of the error in such approximation (in the subdiagonal case) lies between 1/2 and 1/5.828. Furthermore, the approximants they employed to derive the upper bounds had denominators of the form $(1+x/n)^n$. In 1977 it was conjectured that the best uniform approximation to e^{-x} with negative poles has denominator of the form $(1+\alpha x/n)^n$ (that is, all the poles coalesce). This conjecture was made, in one form or another, by various people including Kaufman and Taylor [5, 11], Lau [6], Norsett and Wolfbrandt [8] and Saff, Schonhage and Varga [11]. We shall prove this conjecture.

In [5, 11] it was suggested that the optimal choice of α in the denominator might be $\alpha = 1$. Andersson [1] proved this to be false. He showed $\alpha = \sqrt{2}$ is assymptotically the optimal choice for approximations of the form

$$\frac{p_{n-1}(x)}{(1+\alpha x/n)^n}$$

and that the nth root of the error in this case behaves like $1/(\sqrt{2}+1)$. Our proof of the conjecture shows that this is also the correct rate of approximation by rationals with arbitrary real poles. (This might be compared to an error that behaves like $1/3^n$ for reciprocals of polynomials and that is conjectured to behave like $1/9^n$ for unrestricted rationals).

In 1977, Norsett and Wolfbrant [8] examined a Padé type approximation to e^{-x} with negative poles and showed that such approximations have the property that all the poles coalesce. Lau [6], also in 1977, treated approximations to e^{-x} with negative poles and showed, among other things, that if

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the degree of the numerator is zero then the poles exhibit this same behaviour.

Kaufman and Taylor ask whether this phenomenon holds for approximations to any other functions. We shall show that best approximants to x^h on $[0, 1]$ by rational functions with positive poles also have coalescing poles. One should compare this to results in [7], where approximations to x^h with all negative roots and poles are shown to be of the form dx^n and also to results in [2, 9], where best approximations to e^x and x^h with negative poles are shown to have constant denominators. Actually, in the latter cases, it is the positivity of the coefficients not the location of the zeros that is the issue.

2. Approximating e^{-x}

Let Π_n denote the collection of real algebraic polynomials of degree at most n. Let $R_{n,m}$ denote the rational functions with numerators in Π_n and denominators in Π_m. Let $R_{n,m}^{[\alpha,\beta]}$ denote the subset of $R_{n,m}$ whose denominators have all their zeros in the interval $[\alpha, \beta]$. All best approximations will be with respect to the uniform norm $\|\cdot\|$.

THEOREM 1. *Let $0 < a \leqslant \infty$ and let $n \geqslant m-1$. Let p_n/q_m be a best approximant to e^{-x} on $[0, a]$ from $R_{n,m}^{[-\infty, 0]}$. Then all the poles of q_m coalesce (that is, $q_m = (x+\lambda)^{m'}$, $m' \leqslant m$).*

Proof. Suppose that p_n/q_m has k distinct poles. It follows either as an adaptation of the usual existence and characterization results, or by [5], that p_n/q_m exists and that $e^{-x} - p_n(x)/q_m(x)$ has at least $n+k+1$ distinct zeros in $(0, \infty)$ at which $e^{-x} - p_n(x)/q_m(x)$ changes sign. Thus, by perturbing q_m slightly we can find $q_{n+1}^*(x) = c\prod_{i=1}^{n+1}(x+\lambda_i) \in \Pi_{n+1}$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n+1}$, so that $e^{-x} - p_n(x)/q_{n+1}^*(x)$ has $n+k+1$ distinct zeros (with sign changes) in $(0, \infty)$. We write

$$e^{-x} - \frac{p_n(x)}{q_{n+1}^*(x)} = e^{-x} - \sum_{i=1}^{n+1} \frac{a_i}{x+\lambda_i},$$

where the a_i are real and note that

$$\sum_{i=1}^{n+1} \frac{a_i}{x+\lambda_i} = \int_0^\infty \left(\sum_{i=1}^{n+1} a_i e^{-\lambda_i t}\right) e^{-xt}\, dt.$$

Also, we may approximate e^{-x} on a finite interval $[0, K]$ by

$$g_\delta(x) = \int_0^\infty f_\delta(t)\, e^{-xt}\, dt,$$

where

$$f_\delta(t) = \frac{1}{\delta}, \qquad t \in [1, 1+\delta],$$
$$= 0, \qquad \text{otherwise.}$$

It is now apparent that for suitably chosen δ and K, we can find a piecewise linear continuous function f (approximating f_δ) so that

$$\int_0^\infty \left(f(t) - \sum_{i=1}^{n+1} a_i e^{-\lambda_i t} \right) e^{-xt} \, dt \tag{1}$$

has $n+k+1$ distinct zeros (with sign change) in $(0, \infty)$ and so that

$$f(t) - \sum_{i=1}^{n+1} a_i e^{-\lambda_i t} \tag{2}$$

has at most $n+2$ zeros in $(0, \infty)$. The f we choose to satisfy the above is zero except in a small neighbourhood of 1. Since $\sum_{i=1}^{n+1} a_i e^{-\lambda_i t}$ can have at most n positive zeros we see that we can choose f so that (2) is also satisfied. However, the Laplace transform is variation diminishing (see [4, p. 21]) in the sense that the transform of a function can have no more zeros than the function has. In particular, comparing (1) and (2) above shows that k equals zero or one. ∎

One can prove the result of Norsett and Wolfbrant mentioned in the introduction in a similar fashion.

3. Approximating x^h

We now focus on approximations to x^h.

THEOREM 2. *Let $0 \leqslant \alpha < \beta$, let $n \geqslant m-1$ and let h be a positive integer. Let p_n/q_m be a best approximant to x^h on $[\alpha, \beta]$ from $R_{n,m}^{[\beta,\infty]}$. Then all the poles of q_m coalesce.*

Proof. We may assume that $h > n$. Suppose that q_m has k distinct roots. Once again, as in the proof of Theorem 1, the existence of such an approximant is assured. Also, we can find $q_{n+1}^* \in \Pi_{n+1}$ with $n+1$ distinct roots in $[\beta, \infty)$ so that $x^h - p_n(x)/q_{n+1}^*(x)$ has $n+k+1$ zeros in (α, β). Let

$0 < \lambda_0 < \cdots < \lambda_n < 1/\beta$ be the reciprocals of the roots of q_{n+1}. Then there exist real a_i so that

$$\frac{p_n(x)}{q^*_{n+1}(x)} = \sum_{i=0}^{n} \frac{a_i}{1-\lambda_i x} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} a_i \lambda_i^k \right) x^k.$$

We show that

$$\frac{p_n(x)}{q^*_{n+1}(x)} - x^h$$

has at most $n+2$ zeros in (α, β) by showing that

$$\sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} a_i \lambda_i^k \right) x^k$$

has at most $n+1$ sign changes and then appealing to Descartes' rule of signs. This, of course, shows that k is zero or one.

Suppose that

$$\sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} a_i \lambda_i^k \right) x^k$$

has at least $n+2$ sign changes. Then we can find $m_0 < m_1 < \cdots < m_{n+1}$ so that

$$\begin{pmatrix} \lambda_0^{m_0}, & \lambda_1^{m_0}, & \ldots, & \lambda_n^{m_0}, & \zeta^{m_0} \\ \vdots & \vdots & & & \vdots \\ \lambda_0^{m_{n+1}}, & \lambda_1^{m_{n+1}}, & \ldots, & \lambda_n^{m_{n+1}}, & \zeta^{m_{n+1}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \end{pmatrix} = \pm \begin{pmatrix} b_0 \\ -b_1 \\ +b_2 \\ \vdots \\ (-1)^{n+1} b_{n+1} \end{pmatrix}, (*)$$

where ζ is any real number and the b_i are all strictly positive. Choose $\zeta > \lambda_n$. Since every minor of the above Vandermonde matrix has positive determinant the inverse of the above matrix has all non-zero entries and the sign of the (i, j) entry of the inverse is $(-1)^{i+j}$. Thus, when we multiply the inverse of the above matrix by $(b_0, -b_1, b_2, \ldots, (-1)^{n+1} b_{n+1})$ we get a vector with all non-zero entries. This contradicts $(*)$ and completes the proof. ∎

Theorem 1 can be deduced from Theorem 2 by a limiting argument: set $1/(1 + y/h) = x$ in the approximation to x^h and let h tend to infinity.

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